ENUMERATING VISIBLE TILINGS

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Abstract. To *tile* a square A is to completely fill it with smaller square(s) that have no interior points in common. Such a tiling is said to be *visible* if each one of the tiles has at least one edge that lies on the perimeter of the square A into which they are embedded. It is relatively simple to show that for any natural number $k \geq 6$, there exists a visible tiling of a square with k smaller squares. We further show that for any even natural number $k \geq 6$, there are at least $2^{\frac{(k-6)}{2}}$ distinct visible tilings of a square.

Roughly 30 years ago, Erdős and Soifer published an article [1] entitled "Squares in a Square." This work was based on a previous question Erdős posed regarding squares packed into a square [2]. Much later, in [3], Burt, Staton, and Tyler addressed questions regarding visible tilings.

In this paper, we consider more questions regarding visible tilings of a square. While the existence of visible tilings is addressed in [3], we aim to establish bounds for the number of visible tilings of a square given certain conditions.

Definition 1. To **tile** a square is to completely fill it with square(s), in such a way that none of those squares have any interior points in common.

Figure 1 illustrates examples of tilings with 11, 12, and 20 squares.

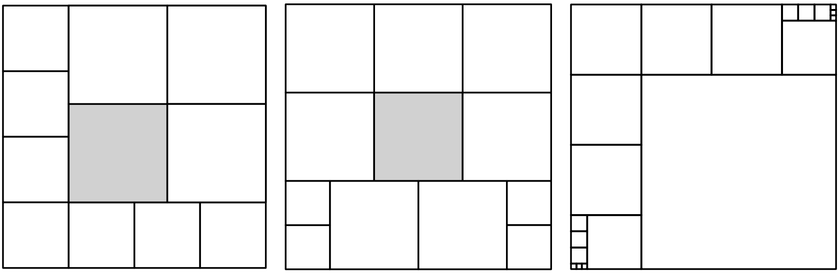

Figure 1: Tilings with $11, 12$, and 20 squares

The following definition for visible tilings is specifically for squares (paraphrased from [3]).

Definition 2. A **visible tiling** of a square is a tiling in which each tile has at least one face which is contained in a face of the larger square.

In Figure 1, note that the first two tilings are not visible (invisible tiles are shaded) but the third tiling is visible.

When dealing with squares specifically, it is relatively simple to show that for any $k \geq 6$, there exists a visible tiling with k squares.

Theorem 1. *For any natural number $k \geq 6$, there exists a visible tiling with k squares.*

Proof. We will consider two cases: one in which k is even, and one in which k is odd.

If k is even, arrange a horizontal series of $\frac{k}{2}$ squares, each of size 1×1, along the top face of the larger square. Then vertically, under the right-most square, create a series of $\frac{k}{2} - 1$ additional squares, each of size 1×1. Then, fill the remaining area with a square of size $\left(\frac{k}{2} - 1\right) \times \left(\frac{k}{2} - 1\right)$.

If k is odd, create a tiling with $k - 3$ squares using the technique described above. Then, take the left-most top square and cut it into four smaller squares, each of size $\frac{1}{2} \times \frac{1}{2}$. Take the two new left-most squares in the top row and move them to the other side of the row, sliding the rest of the squares in the top row to the left $\frac{1}{2}$ a unit. This creates a visible tiling with k squares. $\square$

Figure 2 contains diagrams illustrating this process generating constructions for $k = 26$ and $k = 17$.

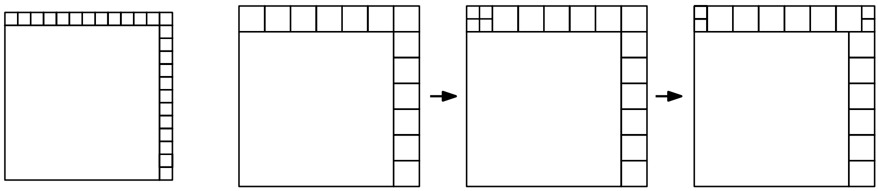

FIGURE 2: Visible tilings with 26 and 17 squares

When considering visible tilings, the previous threshold is interesting but trivial. One question that may be raised, however, is how many different visible tilings exist for any given natural number k? It turns out that when $k = 26$, there are actually well over $1,000$ distinct tilings that we can construct. Enumerating some of these tilings is what we will now discuss.

Before moving forward, we will quickly address how we consider tilings to be distinct for the purposes of this paper.

Definition 3. Tilings will be considered **non-distinct** if they are constructed of the same numbers of squares, and the quantity of all similarly sized squares is the same. Otherwise, the constructions will be **distinct**.

Figure 3 illustrates three tilings that are non-distinct.

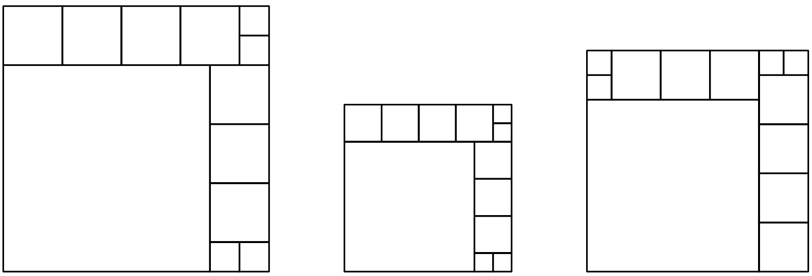

FIGURE 3: An example of non-distinct visible tilings

While it is an interesting question to ask how many different ways visible tilings may be arranged (as in Figure 3), this is not our primary goal here.

Now we discuss a technique that generates visible tilings of a square, each of which is distinct.

For these constructions we will consider tilings with an even number, $k \geq 6$, of squares. First we subtract 2 from k, and then divide the result by 2. We will call this number r, so $r = \frac{k-2}{2}$. Now, we express r as the sum of 1 to $r-1$ natural numbers, where order does matter. The only condition is that the first number in the sequence must be at least 2. We will call the summands used x_i and the total quantity of summands used n, so $r = \sum_{i=1}^{n} x_i$, where $x_1 \geq 2$ and $1 \leq n \leq (r-1)$. Additionally, the x_i square(s) composing the i^{th} term of this sequence will have size s_i.

Now, starting in the top left corner of the larger square, we arrange x_1 squares of size 1 horizontally. We assign $s_1 = 1$. Next, we arrange x_2 squares, each of size $s_2 = (x_1)(s_1) = (x_1)(1)$, vertically below the first row of squares. We then arrange a row of x_3 squares, each of size $s_3 = s_1 + (x_2)(s_2)$, horizontally to the right of these squares. We continue this process, alternating between vertical and horizontal placement, noting that $s_i = (x_{i-1})(s_{i-1}) + (s_{i-2})$, until we have placed the final x_n squares, each of size $s_n = (x_{n-1})(s_{n-1}) + (s_{n-2})$. We duplicate this process starting at the bottom right, initially arranging the squares vertically upwards. This procedure will generate two symmetric rectangles whose corners meet. We complete the process by inserting two final squares, one of size $(x_n)(s_n)+(s_{n-1})$ and another of size s_n, in the empty corners.

Thus, the area of our visibly tiled square will be:

$$2\left(\sum_{i=1}^{n}\left(x_i(s_i)^2\right)\right)+\Big((x_n)(s_n)+(s_{n-1})\Big)^2+(s_n)^2$$

Examples of this process are shown in Figure 4 for $k=16$, where the sequences of numbers are $r=2+2+3$, $r=4+2+1$, $r=3+1+1+2$, and $r=2+1+1+1+2$.

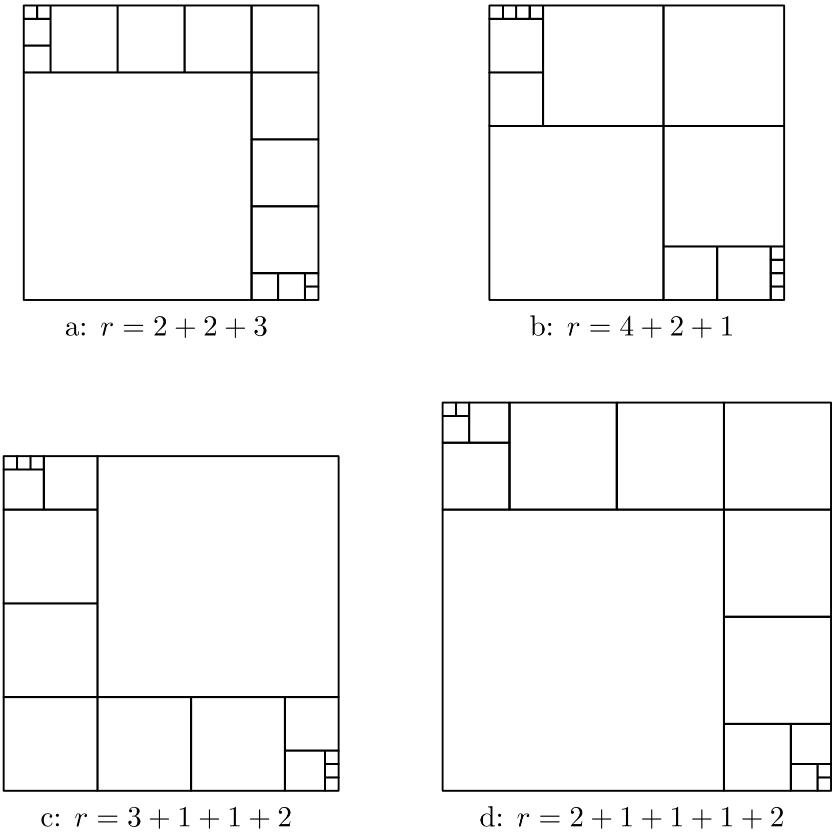

FIGURE 4: Various visible tilings composed of 16 squares

Two notable constructions are generated when $n = 1$ and when $n = r - 1$. Illustrations for these arrangements of squares are shown below for $k = 16$.

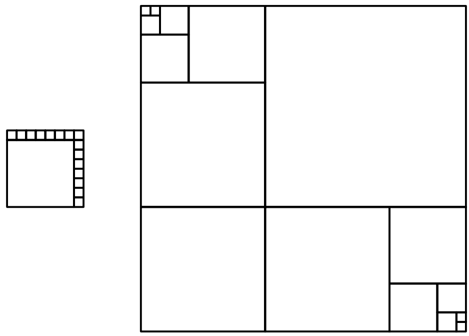

FIGURE 5: Visible tilings with 16 squares when $n = 1$ and $n = r - 1$

Note that when $n = 1$ we have the same type of tiling as in the left-most square in Figure 2. When $n = r - 1$, we have a far more interesting construction. The corner rectangles are formed from two squares of size 1, and then squares of sizes 2, 3, 5, 8, and 13, with the largest corner square having size 21. We appropriately call this tiling a Fibonacci tiling. It is also interesting to note that as k increases to consecutive even values, the side lengths of the tiled squares produced are consecutive Fibonacci numbers. Several of these values are shown below.

k	6	8	10	12	14	16	18	20	22	24	26
Tiled square side length	3	5	8	13	21	34	55	89	144	233	377

TABLE 1: Side lengths of tiled squares for values of k of Fibonacci tilings

An additional point to address is why we require the first number, x_1, in the sequence associated with r to be more than 1. If we let $x_1 = 1$, then s_2 would also be equal to 1. Proceeding this way would simply give us the same construction as if x_1 were at least 2.

We will now use this construction technique to generate lower bounds for the number of distinct tilings for even numbers k when $k \geq 6$.

Theorem 2. *For any even natural number $k \geq 6$, the number of distinct visible tilings of a square with k tiles is at least $2^{\frac{(k-6)}{2}}$.*

Proof. We will prove this lower bound by calculating the number of distinct tilings for each even value k that can be generated using our previously described construction technique for visible tilings. Recall that for k we will generate a sequence of n natural numbers whose sum is $r = \frac{k-2}{2} = \sum_{i=1}^{n} x_i$, where $x_1 \geq 2$.

There is only $\binom{r-2}{0} = 1$ way to express r as the sum of $r-1$ natural numbers when $x_1 \geq 2$. Next, there are $\binom{r-2}{1} = r-2$ ways of expressing r as the sum of exactly $r-2$ natural numbers when $x_1 \geq 2$. Continuing this process, for $x_1 \geq 2$, we ultimately have $\binom{r-2}{r-3} = r-2$ ways to express r as the sum of two natural numbers and $\binom{r-2}{r-2} = 1$ way to express r as the sum of a single natural number.

(To help understand why this is the case, consider the sequence of numbers $2_1 + 1_2 + 1_3 + ... + 1_{r-2} + 1_{r-1}$. This sequence has $r-2$ addition symbols. Thus, there are $\binom{r-2}{p}$ ways to contract p of these symbols, with each resulting in a distinct arrangement of summands.)

Summarizing these results, we have:

$$\binom{r-2}{0} + \binom{r-2}{1} + \binom{r-2}{2} + \cdots + \binom{r-2}{r-3} + \binom{r-2}{r-2} = 2^{(r-2)}$$

total different ways to express r. Since $r = \frac{k-2}{2}$, we will have at least $2^{\left(\frac{k-2}{2}-2\right)} = 2^{\left(\frac{k-2-4}{2}\right)} = 2^{\left(\frac{k-6}{2}\right)}$ distinct ways to visibly tile a square with k smaller squares.

$\square$

The 8 constructions generated by this process when $k = 12$ are shown below, with the last being a Fibonacci tiling. Each row represents increasing values of n, where $1 \leq n \leq 4$.

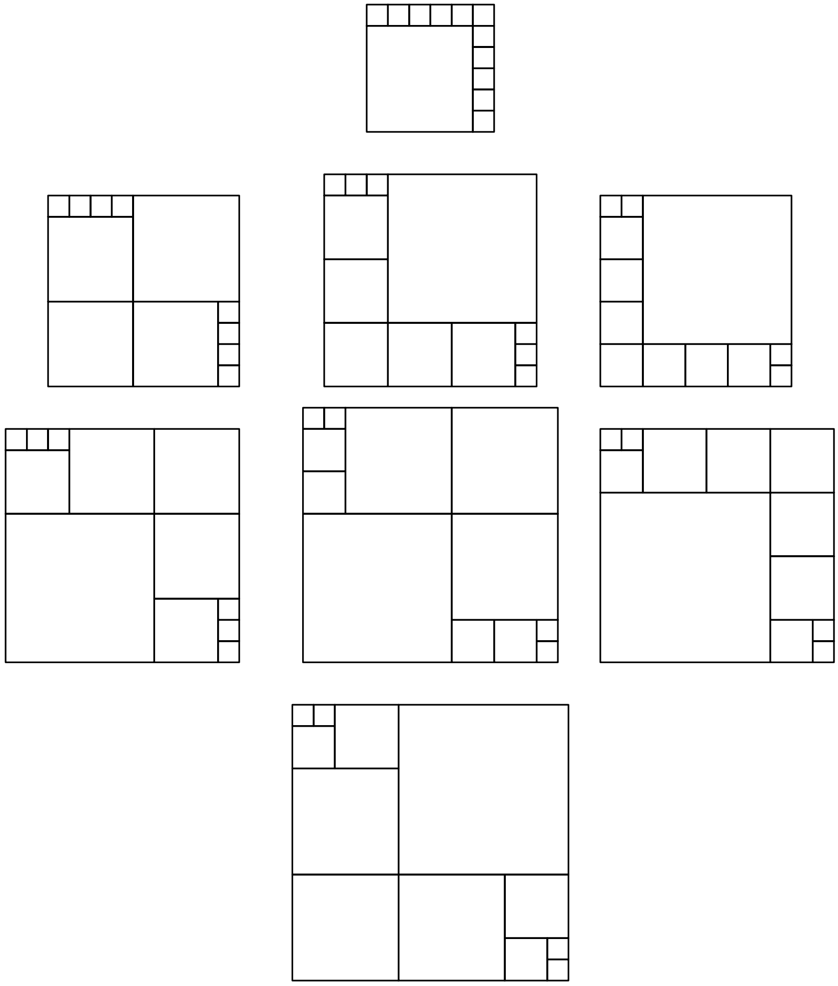

FIGURE 6: The 8 visible tilings generated using our technique for $k = 12$

While this construction technique does generate a large quantity of distinct tilings, there are visible tilings with larger values of k which are not generated by this method. Three other visible tilings for $k = 12$, not generated by this method, are shown below.

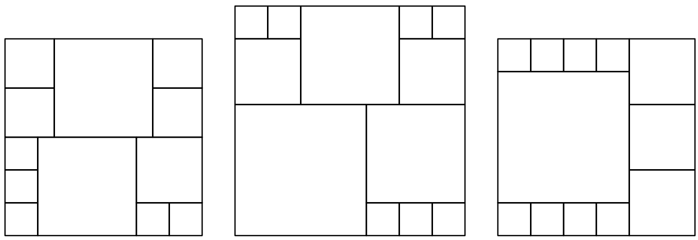

FIGURE 7: Distinct visible tilings for $k = 12$ not generated using our technique

Many further questions can be posed regarding visible tilings of squares and cubes. In [3], several ideas are addressed that also involve tiling hypercubes with dimensions higher than two.

We would like to dedicate this work to our late friend and colleague, Dr. Bill Staton.

References

[1] Erdős, P., Soifer, A., Squares in Square. Geombinatorics, volume IV, issue 4 (1995), 110-114.

[2] Erdős, P., Some of my favorite problems in number theory. Comb. Week. Resenhas 2 (1995), no. 2, 165-186.

[3] Burt, J., Staton, W., Tyler, B., On Visible Tilings. Geombinatorics, volume XXV, issue 3 (2016), 103-112.