Enumerating Maximal Independent Sets in Grid-like Graphs

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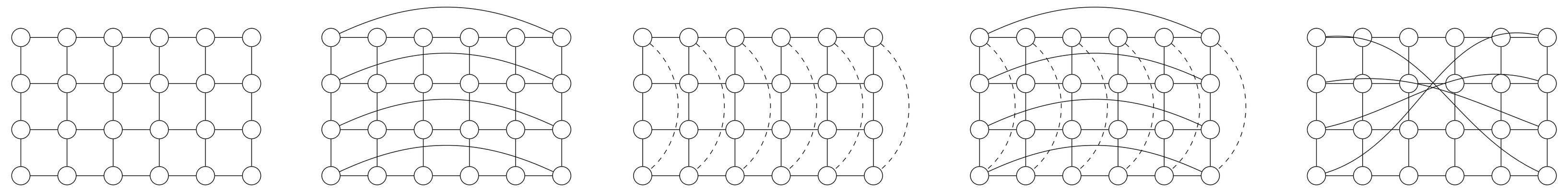
Abstract

We study the problem of enumerating *maximal independent sets* (MIS's) on grid graphs embedded on different surfaces. We approach enumeration by constructing MIS's one column at a time via *states*, which specify the set of vertices to be included in a column of the MIS as well as the information about the state coming before it that is relevant to determining which states can come after it. We form the *map digraph* describing how states can follow each other, which transforms the problem into finding the number of fixed-length walks with certain starting and ending states specified by the *ticket digraph*, which varies with the identifications of the surface on which the grid is embedded. Using the properties of the map digraph, we prove a characterization of the exponential growth of the number of such walks. Additionally, we prove a general result about the ratio of MIS's on two graphs with the same local structure, differentiated by a permutation of ticket destinations determined by a graph automorphism of the map digraph.

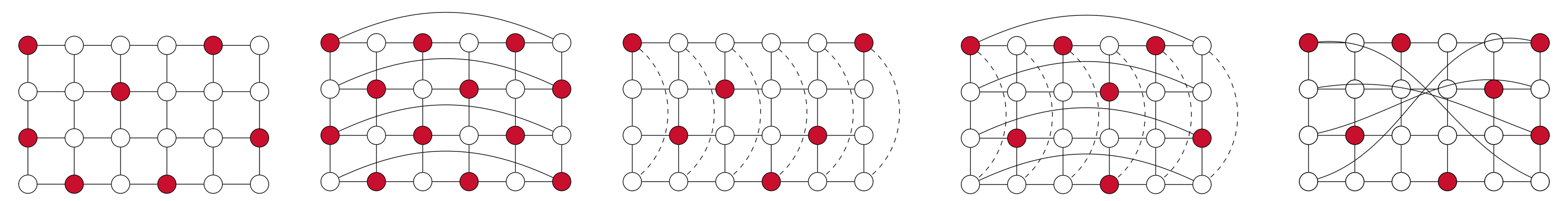
Grid Graphs and MIS's

We study *grid-like* graphs with m rows and n columns: for example, the flat grid, the thin and fat cylinders, the torus, and the Möbius strip. We denote the graph Cartesian product by $\square$:



$G_{m\times n} = P_m\square P_n$ $FC_{m\times n} = P_m\square C_n$ $TC_{m\times n} = C_m\square P_n$ $T_{m\times n} = C_m\square C_n$ $M_{m\times n}$

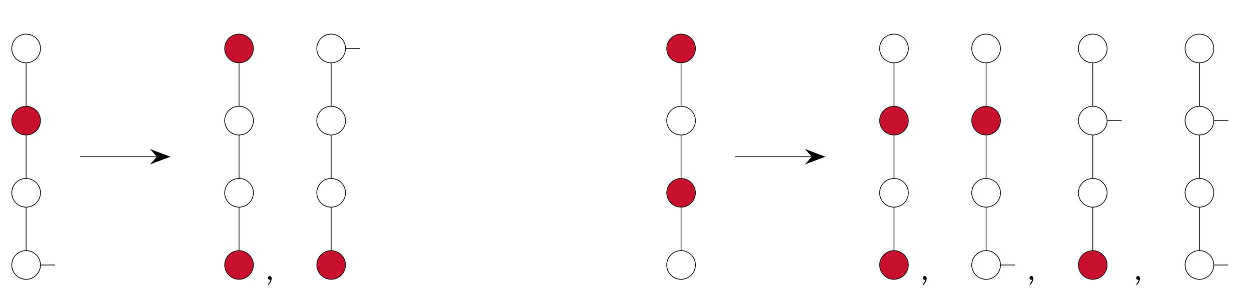
An *independent set* I is a set of vertices where no vertices in I are adjacent. We say I is a *maximal independent set* (*MIS*) if I is not a proper subset of any independent set. Thus, every vertex is *covered*, i.e., is in I or has a neighbor in I. Below are examples of MIS's in our grid-like graphs:



Flat Grid Fat Cylinder Thin Cylinder Torus Möbius Strip

How to Build an MIS

We can think of constructing MIS's column by column: we begin with a column that has a (valid) independent set and then concatenate $n-1$ columns to the right. We say that each column is in a *state*, with an independent set of vertices that contributes to the MIS and a *deficit*, the set of vertices that must be covered by a column following it.



Here, we draw states with red vertices denoting the independent set and a tick mark denoting the deficit. Shown are two example states with $H = P_4$ with the states that can follow each.

Map Digraph

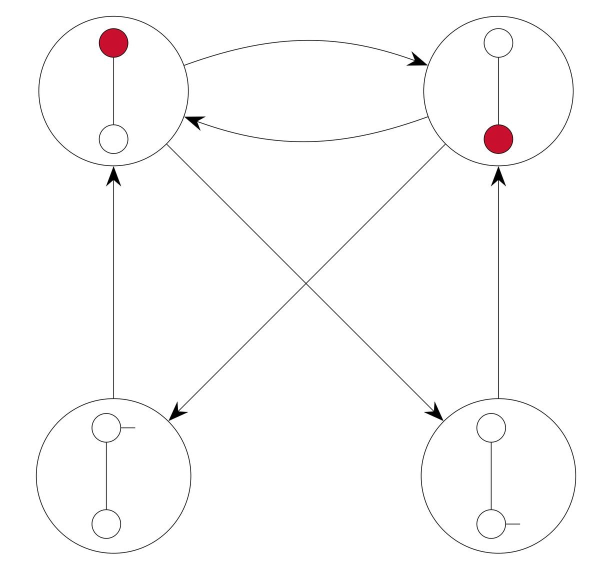


Figure 1. P_2 Map Digraph

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

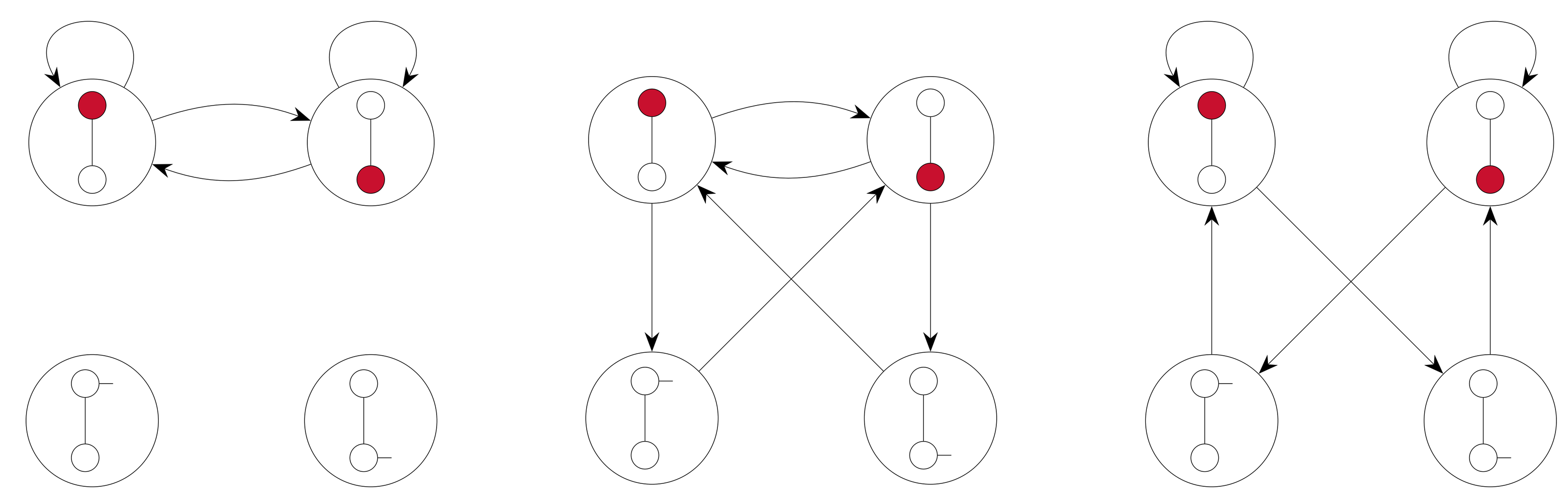
Figure 2. Transfer Matrix

The *transfer matrix* for a local structure is the (transpose of the) adjacency matrix of its map digraph. We record an arc from state i to state j with a 1 in the (j, i)-th entry.

We found that for any local structure, for any $n \geq 4$ and for any states i, j, there exists a walk of length n in the map digraph from state i to state j. This means that A^n is strictly positive, where A is the relevant transfer matrix, i.e., A is primitive.

Ticket Digraph

The *ticket digraph* encodes the starts and ends of MIS's. We include an arc from state i to state j if an MIS can have its first column in state i and its last column in state j given the global structure.



Global Path Structure Global Cyclic Structure Global Möbius Structure

Metro Pairs and Travel Sequences

Definition

A *metro pair* is an ordered pair (M, T) of directed graphs that share a vertex set such that the adjacency matrix of M is primitive. We call M the *map digraph* and T the *ticket digraph*. The *travel sequence* of a metro pair (M, T) is a sequence τ defined by

$$\tau(n) = A_T \bullet A_M^n,$$

where A_M and A_T are the (transposes of the) adjacency matrices of M and T, and $\bullet$ represents the dot product as vectors.

Given a metro pair, the travel sequence counts walks of length n along the map digraph whose endpoints correspond to an arc in the ticket digraph. Note that these exactly correspond to MIS's.

Growth of Entries

Theorem 1

Let (M, T) be a metro pair with associated matrices A_M and A_T and travel sequence τ. Then τ obeys the linear recurrence relation given by the characteristic polynomial of A_M. Let $r > 1$ be the Perron-Frobenius eigenvalue of A_M with left and right eigenspaces $\langle \vec{w}^\top \rangle$ and $\langle \vec{v} \rangle$ respectively. Then

$$\lim_{n\to\infty} \frac{\tau(n)}{r^n} = A_T \bullet \frac{\vec{v}\vec{w}^\top}{\vec{w}^\top \vec{v}} > 0.$$

Comparing Shapes

As a corollary of Theorem 1, if two metro pairs have the same map digraph, then the ratio of their travel sequences converges. Further, it allows us to compute what this limit is, e.g.

$$\lim_{n\to\infty} \frac{|\operatorname{MIS}(G_{2\times n})|}{|\operatorname{MIS}(FC_{2\times n})|} = \frac{2}{\sqrt{5}}.$$

In some special cases, we know that this limit is 1:

Theorem 2

Let (M, T) be a metro pair with travel sequence τ. Let ϕ be a digraph automorphism on M. Let (M, T') be a metro pair with travel sequence τ' such that $(u, v) \in E(T)$ if and only if $(u, \phi(v)) \in E(T')$. Then

$$\lim_{n\to\infty} \frac{\tau'(n)}{\tau(n)} = 1.$$

Any graph automorphism on the local structure of a grid induces a digraph automorphism on its map digraph. As a result, we can permute the destinations of the tickets of a metro pair according to a graph automorphism. For example, taking local path structure, we can consider walks from states to their vertical flips as opposed to walks from states to themselves, and get for any m that

$$\lim_{n\to\infty} \frac{|\operatorname{MIS}(M_{m\times n})|}{|\operatorname{MIS}(FC_{m\times n})|} = 1.$$

A similar result holds for ratios between the MIS's of tori, twisted tori, and Klein bottles.

References

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